Operator preconditioning for EMI equations

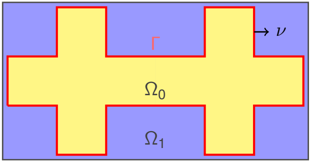
Miroslav Kuchta
simula

MICROCARD

3rd MICROCARD Workshop

5th July 2023, Strasbourg

Operator preconditioning in a nutshell

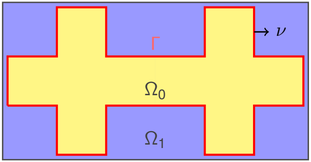


$$
\begin{aligned}
-\nabla \cdot (K_0 \nabla u_0) &= 0 && \text{in } \Omega_0, \\
-\nabla \cdot (K_1 \nabla u_1) &= 0 && \text{in } \Omega_1, \\
-K_0 \nabla u_0 \cdot \nu + K_1 \nabla u_1 \cdot \nu &= 0 && \text{on } \Gamma, \\
u_0 - u_1 + \varepsilon K_0 \nabla u_0 \cdot \nu &= f && \text{on } \Gamma.
\end{aligned}
$$

Goal: Construct scalable and K_i, ε-robust solvers for linear systems

Find $u_h \in W_h$ such that $\mathcal{A}_h u_h = b_h$ $\approx R^n : \mathbf{Au} = \mathbf{b}$

Operator preconditioning in a nutshell

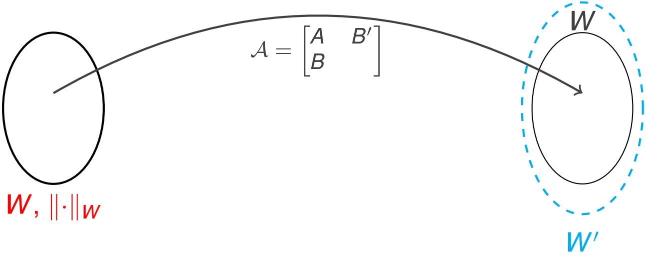


$$
\begin{aligned}
-\nabla\cdot(K_0\nabla u_0) &= 0 && \text{in } \Omega_0,\\
-\nabla\cdot(K_1\nabla u_1) &= 0 && \text{in } \Omega_1,\\
-K_0\nabla u_0\cdot\nu + K_1\nabla u_1\cdot\nu &= 0 && \text{on } \Gamma,\\
u_0 - u_1 + \varepsilon K_0\nabla u_0\cdot\nu &= f && \text{on } \Gamma.
\end{aligned}
$$

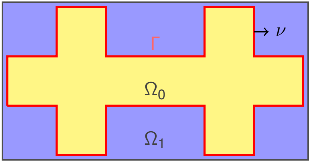
Goal: Construct scalable and K_i, ε-robust solvers for linear systems

Find $u_h \in W_h$ such that $\mathcal{A}_h u_h = b_h$ $\qquad \approx R^n : \mathbf{Au} = \mathbf{b}$

Construction reflects origin of $\mathcal{A}_h$ in a continuous problem, $W_h \approx W$



Operator preconditioning in a nutshell

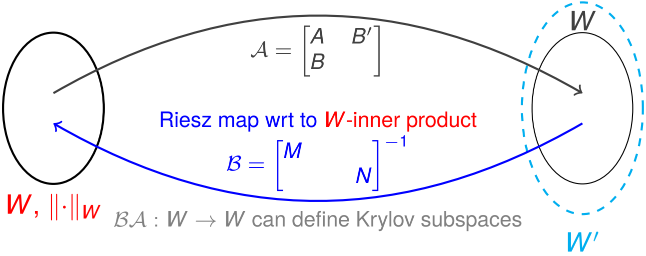


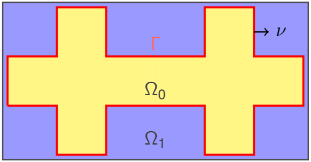
$$
\begin{aligned}
-\nabla \cdot (K_0 \nabla u_0) &= 0 && \text{in } \Omega_0,\\
-\nabla \cdot (K_1 \nabla u_1) &= 0 && \text{in } \Omega_1,\\
-K_0 \nabla u_0 \cdot \nu + K_1 \nabla u_1 \cdot \nu &= 0 && \text{on } \Gamma,\\
u_0 - u_1 + \varepsilon K_0 \nabla u_0 \cdot \nu &= f && \text{on } \Gamma.
\end{aligned}
$$

Goal: Construct scalable and K_i, ε-robust solvers for linear systems

Find $u_h \in W_h$ such that $\mathcal{A}_h u_h = b_h$ $\approx R^n : \mathbf{Au} = \mathbf{b}$

Construction reflects origin of $\mathcal{A}_h$ in a continuous problem, $W_h \approx W$



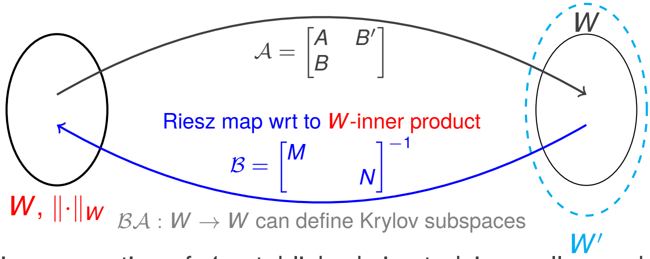


$$
\begin{aligned}
-\nabla\cdot(K_0\nabla u_0) &= 0 && \text{in } \Omega_0,\\
-\nabla\cdot(K_1\nabla u_1) &= 0 && \text{in } \Omega_1,\\
-K_0\nabla u_0\cdot\nu + K_1\nabla u_1\cdot\nu &= 0 && \text{on } \Gamma,\\
u_0 - u_1 + \varepsilon K_0\nabla u_0\cdot\nu &= f && \text{on } \Gamma.
\end{aligned}
$$

Goal: Construct scalable and K_i, ε-robust solvers for linear systems

Find $u_h \in W_h$ such that $\mathcal{A}_h u_h = b_h$ $\quad\approx R^n : \mathbf{Au} = \mathbf{b}$

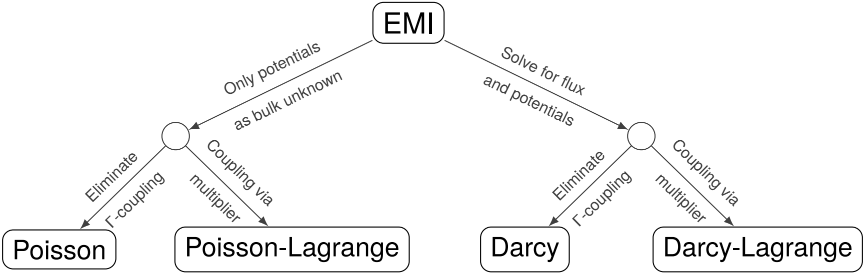
Construction reflects origin of $\mathcal{A}_h$ in a continuous problem, $W_h \approx W$

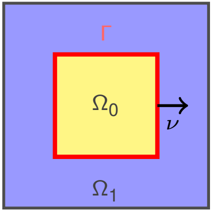


- Mapping properties of $\mathcal{A}$ established via studying well-posedness
- Stable discretization yields $\text{cond}(\mathcal{B}_h\mathcal{A}_h) \le C \ne C(h, \varepsilon, K_i)$

Mardal, K., Winther, R. (2011). Preconditioning discretizations of systems of PDEs. NUMER LINEAR ALGEBR. Kuchta, M., Mardal, K. A. (2020). Iterative solvers for EMI models. Modeling Excitable Tissue The EMI Framework, 70.

The many ways of EMI





$$
\begin{aligned}
-\nabla\cdot(K_0\nabla u_0) &= 0 \quad \text{in } \Omega_0,\\
-\nabla\cdot(K_1\nabla u_1) &= 0 \quad \text{in } \Omega_1,\\
-K_0\nabla u_0\cdot\nu + K_1\nabla u_1\cdot\nu &= 0 \quad \text{on } \Gamma,\\
\underbrace{u_0|_\Gamma - u_1|_\Gamma}_{=[\![u]\!]} + \varepsilon K_0\nabla u_0\cdot\nu &= f \quad \text{on } \Gamma.
\end{aligned}
$$

For simplicity $u_1 = 0$ in $\partial\Omega_1 \setminus \Gamma$

Apply Robin condition in integration by parts

With Γ-discontinous test/trial functions: *Find* $u = [u_0, u_1] \in W$ *such that*

$$\int_{\Omega_0} K_0\nabla u_0\cdot\nabla v_0 + \int_{\Omega_1} K_1\nabla u_1\cdot\nabla v_1 + \varepsilon^{-1}\int_\Gamma [\![u]\!][\![v]\!] = \varepsilon^{-1}\int_\Gamma f[\![v]\!] \quad \forall v\in W$$

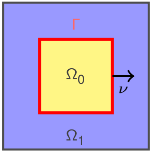
Equivalent operator equation $\mathcal{A}u = b$ in W'

$$\underbrace{\begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 & -\varepsilon^{-1}T_0'T_1 \\ -\varepsilon^{-1}T_1'T_0 & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = b$$

Well-posedness in $W = H^1(\Omega_0) \times H^1_0(\Omega_1)$ by Lax-Milgram theorem

Do standard norms yield ε-robust preconditioners?

Poisson-like formulation with canonical Riesz map



$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 & -\varepsilon^{-1}T_0'T_1 \\ -\varepsilon^{-1}T_1'T_0 & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}$$

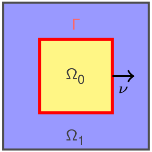
Norm of W due (almost) standard inner-product

$$\|u\|_W^2 = \int_{\Omega_0} K_0\nabla u_0 \cdot \nabla v_0 + \int_{\Omega_0} K_0 u_0 v_0 + \int_{\Omega_1} K_1\nabla u_1 \cdot \nabla v_1$$

Choice of inner product induces a concrete Riesz operator

$$\mathcal{B} = \begin{bmatrix} -K_0\Delta_0 + K_0 I_0 & \\ & -K_1\Delta_1 \end{bmatrix}^{-1}$$

Poisson-like formulation with canonical Riesz map



$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 & -\varepsilon^{-1}T_0'T_1 \\ -\varepsilon^{-1}T_1'T_0 & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}$$

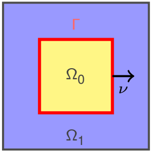
Norm of W due (almost) standard inner-product

$$\|u\|_W^2 = \int_{\Omega_0} K_0 \nabla u_0 \cdot \nabla v_0 + \int_{\Omega_0} K_0 u_0 v_0 + \int_{\Omega_1} K_1 \nabla u_1 \cdot \nabla v_1$$

Choice of inner product induces a concrete Riesz operator $\|u\|_W = \sqrt{\langle \mathcal{B}^{-1}u, u\rangle}$

$C(\varepsilon) \neq C \leq \inf_v \frac{\langle Av, v\rangle}{\|v\|_W}$

$$\mathcal{B} = \begin{bmatrix} -K_0\Delta_0 + K_0 I_0 & \\ & -K_1\Delta_1 \end{bmatrix}^{-1}$$



$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 & -\varepsilon^{-1}T_0'T_1 \\ -\varepsilon^{-1}T_1'T_0 & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}$$

Norm of W due (almost) standard inner-product

$$\|u\|_W^2 = \int_{\Omega_0} K_0 \nabla u_0 \cdot \nabla v_0 + \int_{\Omega_0} K_0 u_0 v_0 + \int_{\Omega_1} K_1 \nabla u_1 \cdot \nabla v_1$$

Choice of inner product induces a concrete Riesz operator $\|u\|_W = \sqrt{\langle \mathcal{B}^{-1}u, u\rangle}$

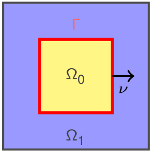
$C(\varepsilon) \neq C \leq \inf_v \frac{\langle Av, v\rangle}{\|v\|_W}$

$$\mathcal{B} = \begin{bmatrix} -K_0\Delta_0 + K_0 I_0 & \\ & -K_1\Delta_1 \end{bmatrix}^{-1}$$

$\sup_v \sup_u \frac{\langle Au, v\rangle}{\|u\|_W \|v\|_W} \leq C = C(\varepsilon)$

Poisson-like formulation with canonical Riesz map

$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 & -\varepsilon^{-1}T_0'T_1 \\ -\varepsilon^{-1}T_1'T_0 & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}$$



Norm of W due (almost) standard inner-product

$$\|u\|_W^2 = \int_{\Omega_0} K_0 \nabla u_0 \cdot \nabla v_0 + \int_{\Omega_0} K_0 u_0 v_0 + \int_{\Omega_1} K_1 \nabla u_1 \cdot \nabla v_1$$

Choice of inner product induces a concrete Riesz operator

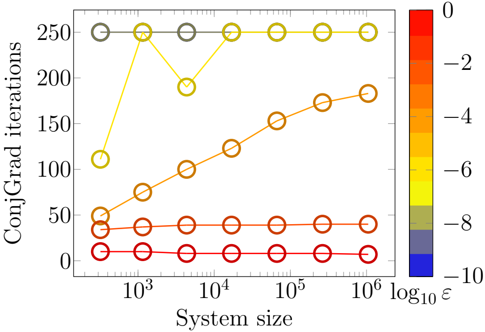
$\|u\|_W = \sqrt{\langle \mathcal{B}^{-1}u, u\rangle}$

$C(\varepsilon) \neq C \leq \inf_v \frac{\langle Av, v\rangle}{\|v\|_W}$

$$\mathcal{B} = \begin{bmatrix} -K_0\Delta_0 + K_0 I_0 & \\ & -K_1\Delta_1 \end{bmatrix}^{-1}$$

$\sup_v \sup_u \frac{\langle Au, v\rangle}{\|u\|_W \|v\|_W} \leq C = C(\varepsilon)$

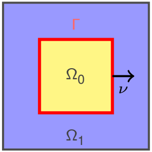
$\mathbb{P}_1$-$\mathbb{P}_1$ elements
$K_0 = 0$, $K_1 = 2$
relative tolerance 10^{-10}
maximum iterations 250
preconditioner by LU



Iterations affected by ε

Poisson-like formulation with “Jacobi” Riesz map

$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1} T_0' T_0 & -\varepsilon^{-1} T_0' T_1 \\ -\varepsilon^{-1} T_1' T_0 & -K_1 \Delta_1 + \varepsilon^{-1} T_1' T_1 \end{bmatrix}$$



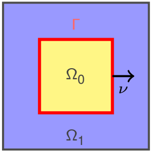
Norm of W due diagonal of $\mathcal{A}$

$$\|u\|_W^2 = \int_{\Omega_0} K_0 \nabla u_0 \cdot \nabla v_0 + \int_\Gamma \varepsilon^{-1} u_0 v_0 + \int_{\Omega_0} K_0 u_0 v_0 + \int_{\Omega_1} K_1 \nabla u_1 \cdot \nabla v_1 + \int_\Gamma \varepsilon^{-1} u_1 v_1$$

Choice of inner product induces Riesz operator

$$\mathcal{B} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1} T_0' T_0 + K_0 I_0 & \\ & -K_1 \Delta_1 + \varepsilon^{-1} T_1' T_1 \end{bmatrix}^{-1}$$

$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 & -\varepsilon^{-1}T_0'T_1 \\ -\varepsilon^{-1}T_1'T_0 & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}$$



Norm of W due diagonal of $\mathcal{A}$

$$\|u\|_W^2 = \int_{\Omega_0} K_0 \nabla u_0 \cdot \nabla v_0 + \int_\Gamma \varepsilon^{-1} u_0 v_0 + \int_{\Omega_0} K_0 u_0 v_0 + \int_{\Omega_1} K_1 \nabla u_1 \cdot \nabla v_1 + \int_\Gamma \varepsilon^{-1} u_1 v_1$$

Choice of inner product induces Riesz operator

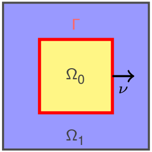
$\|u\|_W = \sqrt{\langle \mathcal{B}^{-1}u, u\rangle}$

$$\mathcal{B} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 + K_0 I_0 & \\ & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}^{-1}$$

$\sup_v \sup_u \frac{\langle Au, v\rangle}{\|u\|_W \|v\|_W} \leq C \neq C(\varepsilon)$

Poisson-like formulation with “Jacobi” Riesz map

$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 & -\varepsilon^{-1}T_0'T_1 \\ -\varepsilon^{-1}T_1'T_0 & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}$$



Norm of W due diagonal of $\mathcal{A}$

$$\|u\|_W^2 = \int_{\Omega_0} K_0 \nabla u_0 \cdot \nabla v_0 + \int_\Gamma \varepsilon^{-1} u_0 v_0 + \int_{\Omega_0} K_0 u_0 v_0 + \int_{\Omega_1} K_1 \nabla u_1 \cdot \nabla v_1 + \int_\Gamma \varepsilon^{-1} u_1 v_1$$

Choice of inner product induces Riesz operator

$\|u\|_W = \sqrt{\langle \mathcal{B}^{-1}u, u\rangle}$

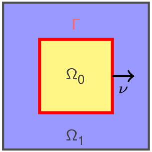
$$\mathcal{B} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 + K_0 I_0 & \\ & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}^{-1}$$

$C(\varepsilon) = C \leq \inf_v \frac{\langle Av, v\rangle}{\|v\|_W}$

$\sup_v \sup_u \frac{\langle Au, v\rangle}{\|u\|_W \|v\|_W} \leq C \neq C(\varepsilon)$

Poisson-like formulation with "Jacobi" Riesz map

$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 & -\varepsilon^{-1}T_0'T_1 \\ -\varepsilon^{-1}T_1'T_0 & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}$$



Norm of W due diagonal of $\mathcal{A}$

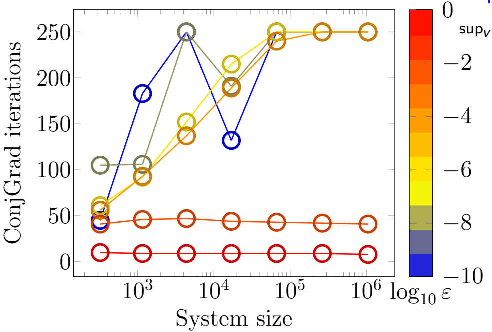
$$\|u\|_W^2 = \int_{\Omega_0} K_0\nabla u_0\cdot\nabla v_0 + \int_\Gamma \varepsilon^{-1}u_0v_0 + \int_{\Omega_0} K_0u_0v_0 + \int_{\Omega_1} K_1\nabla u_1\cdot\nabla v_1 + \int_\Gamma \varepsilon^{-1}u_1v_1$$

Choice of inner product induces Riesz operator

$\|u\|_W = \sqrt{\langle \mathcal{B}^{-1}u, u\rangle}$

$$\mathcal{B} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 + K_0I_0 & \\ & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}^{-1}$$

$C(\varepsilon) = C \le \inf_v \frac{\langle Av, v\rangle}{\|v\|_W}$



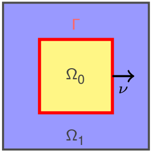
$\sup_v \sup_u \frac{\langle Au, v\rangle}{\|u\|_W\|v\|_W} \le C \ne C(\varepsilon)$

$\mathbb{P}_1$-$\mathbb{P}_1$ elements
$K_0 = 0$, $K_1 = 2$
relative tolerance 10^{-10}
maximum iterations 250
preconditioner by LU

Iterations affected by ε

Poisson-like formulation with "energy" Riesz map

$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 & -\varepsilon^{-1}T_0'T_1 \\ -\varepsilon^{-1}T_1'T_0 & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}$$



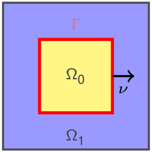
$\mathcal{A}$ induces inner product on W

$$\|u\|_W^2 = \langle \mathcal{A}u, u\rangle \text{ so that } 1\cdot\|u\|_W^2 \leq \langle Au, u\rangle, \quad \langle Au, v\rangle \leq 1\cdot\|u\|_W\|v\|_W$$

The Riesz map preconditioner is then $\mathcal{B} = \mathcal{A}^{-1}$ *Is it easy to invert?*

Poisson-like formulation with “energy” Riesz map

$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 & -\varepsilon^{-1}T_0'T_1 \\ -\varepsilon^{-1}T_1'T_0 & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}$$

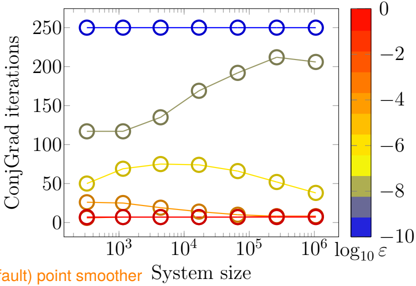


$\mathcal{A}$ induces inner product on W

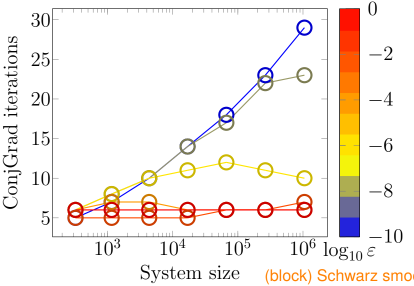
$$\|u\|_W^2 = \langle \mathcal{A}u, u\rangle \text{ so that } 1\cdot\|u\|_W^2 \leq \langle Au, u\rangle, \quad \langle Au, v\rangle \leq 1\cdot\|u\|_W\|v\|_W$$

The Riesz map preconditioner is then $\mathcal{B} = \mathcal{A}^{-1}$ *Is it easy to invert?*

$\mathcal{B}$ by HYPRE BoomerAMG V-cycle



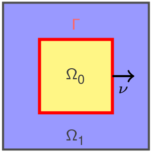
(default) point smoother



(block) Schwarz smoother

Poisson-like formulation with "energy" Riesz map

$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1} T_0' T_0 & -\varepsilon^{-1} T_0' T_1 \\ -\varepsilon^{-1} T_1' T_0 & -K_1\Delta_1 + \varepsilon^{-1} T_1' T_1 \end{bmatrix}$$

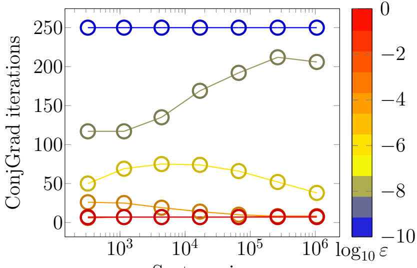


$\mathcal{A}$ induces inner product on W

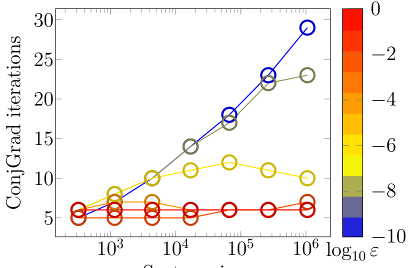
$$\|u\|_W^2 = \langle \mathcal{A}u, u\rangle \text{ so that } 1\cdot\|u\|_W^2 \leq \langle \mathcal{A}u, u\rangle, \quad \langle \mathcal{A}u, v\rangle \leq 1\cdot\|u\|_W\|v\|_W$$

The Riesz map preconditioner is then $\mathcal{B} = \mathcal{A}^{-1}$ *Is it easy to invert?*

$\mathcal{B}$ by HYPRE BoomerAMG V-cycle



(default) point smoother

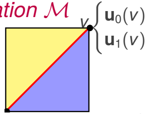


(block) Schwarz smoother

Handling $\varepsilon \ll 1$ requires AMG smoothers capturing kernel of metric perturbation $\mathcal{M}$

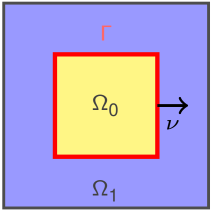
diffusion $= \frac{\text{length scale}^2}{\text{time scale}}$

$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 & \\ & -K_1\Delta_1 \end{bmatrix} + \varepsilon^{-1} \overbrace{\begin{bmatrix} T_0' T_0 & -T_0 T_1 \\ -T_1' T_0 & T_1' T_1 \end{bmatrix}}^{\mathcal{M}}$$



Budisa, A., et al. (2023). AMG methods for metric-perturbed coupled problems. arXiv:2305.06073.

Poisson-Lagrange formulation



$$
\begin{aligned}
-\nabla\cdot(K_0\nabla u_0) &= 0 && \text{in } \Omega_0,\\
-\nabla\cdot(K_1\nabla u_1) &= 0 && \text{in } \Omega_1,\\
-K_0\nabla u_0\cdot\nu + K_1\nabla u_1\cdot\nu &= 0 && \text{on } \Gamma,\\
u_0|_\Gamma - u_1|_\Gamma + \varepsilon\underbrace{K_0\nabla u_0\cdot\nu}_{=-\lambda:\Gamma\to\mathbb{R}} &= f && \text{on } \Gamma.
\end{aligned}
$$

Lagrange multiplier λ enforces Robin

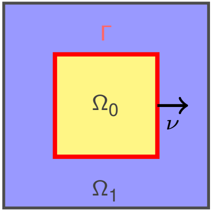
Saddle point problem for $x=[u_0,u_1,\lambda]\in W$ with operator

$$
\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 & & T_0' \\ & -K_1\Delta_1 & -T_1' \\ T_0 & -T_1 & -\varepsilon I_\Gamma \end{bmatrix}
$$

Well-posedness in $W = H^1(\Omega_0)\times H^1_0(\Omega_1)\times H^{-1/2}(\Gamma)$ by Brezzi theory

Are fractional spaces really needed?

Poisson-Lagrange formulation



$$
\begin{aligned}
-\nabla\cdot(K_0\nabla u_0) &= 0 && \text{in } \Omega_0,\\
-\nabla\cdot(K_1\nabla u_1) &= 0 && \text{in } \Omega_1,\\
-K_0\nabla u_0\cdot\nu + K_1\nabla u_1\cdot\nu &= 0 && \text{on } \Gamma,\\
u_0|_\Gamma - u_1|_\Gamma + \varepsilon \underbrace{K_0\nabla u_0\cdot\nu}_{=-\lambda:\Gamma\to\mathbb{R}} &= f && \text{on } \Gamma.
\end{aligned}
$$

Lagrange multiplier λ enforces Robin

Saddle point problem for $x = [u_0, u_1, \lambda] \in W$ with operator

$$
\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 & & T_0' \\ & -K_1\Delta_1 & -T_1' \\ T_0 & -T_1 & -\varepsilon I_\Gamma \end{bmatrix}
$$

Well-posedness in $W = H^1(\Omega_0)\times H^1_0(\Omega_1)\times H^{-1/2}(\Gamma)$ by Brezzi theory

Are fractional spaces really needed?

Assume $H^1(\Omega_0)\times H^1_0(\Omega_1)\times L^2(\Gamma)$ leading to Riez map

$$
\mathcal{B} = \begin{bmatrix} -K_0\tilde{\Delta}_0 & & \\ & -K_1\Delta_1 & \\ & & I_\Gamma \end{bmatrix}^{-1}
$$

For convenience let $\tilde{\Delta}_0 = \Delta_0 - I_0$

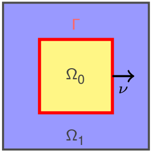
$\mathbb{P}_1$-$\mathbb{P}_1$-$\mathbb{P}_1$ elements, $K_0 = 1$, $K_1 = 2$, *preconditioner by LU*

ε \ h	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
1	23	23	23	22	22
10^{-2}	86	89	91	92	85
10^{-4}	143	183	237	250	250

MinRes iterations are unbounded in ε

Intersection spaces for Poisson-Lagrange formulation

$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 & & T_0' \\ & -K_1\Delta_1 & -T_1' \\ T_0 & -T_1 & -\varepsilon I_\Gamma \end{bmatrix} : W_i \to W_i'$$

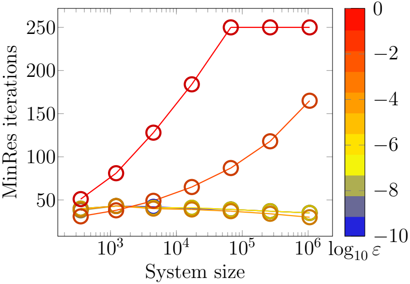


Different function space setting via LM space

$W_0 = H^1 \times H^1_0 \times H^{-1/2}$

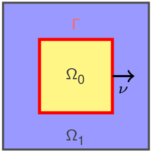
$$\mathcal{B}_0 = \begin{bmatrix} -K_0\tilde{\Delta}_0 & & \\ & -K_1\Delta_1 & \\ & & -\overline{K}^{-1}\Delta_\Gamma^{-1/2} \end{bmatrix}^{-1}$$

$\overline{K} = \frac{K_0 K_1}{K_0 + K_1}$



Intersection spaces for Poisson-Lagrange formulation

$$\mathcal{A} = \begin{bmatrix} -K_0\Delta_0 & & T_0' \\ & -K_1\Delta_1 & -T_1' \\ T_0 & -T_1 & -\varepsilon I_\Gamma \end{bmatrix} : W_i \rightarrow W_i'$$



Different function space setting via LM space

$$W_0 = H^1 \times H_0^1 \times H^{-1/2}$$

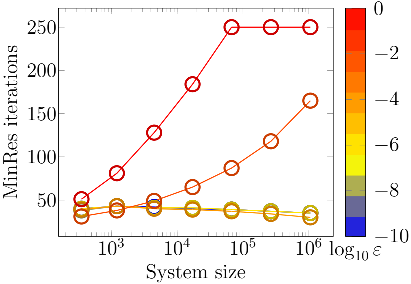
$$\mathcal{B}_0 = \begin{bmatrix} -K_0\tilde{\Delta}_0 & & \\ & -K_1\Delta_1 & \\ & & -\overline{K}^{-1}\Delta_\Gamma^{-1/2} \end{bmatrix}^{-1}$$

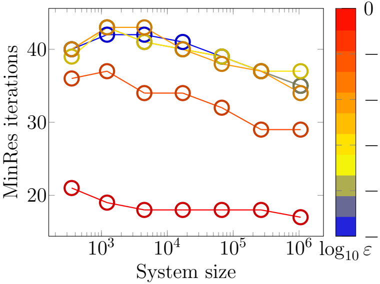
$$W_1 = H^1 \times H_0^1 \times H^{-1/2} \cap \varepsilon^{1/2} I$$

$$\mathcal{B}_1 = \begin{bmatrix} -K_0\tilde{\Delta}_0 & & \\ & -K_1\Delta_1 & \\ & & -\overline{K}^{-1}\Delta_\Gamma^{-1/2} + \varepsilon I_\Gamma \end{bmatrix}^{-1}$$

$\overline{K} = \frac{K_0 K_1}{K_0 + K_1}$

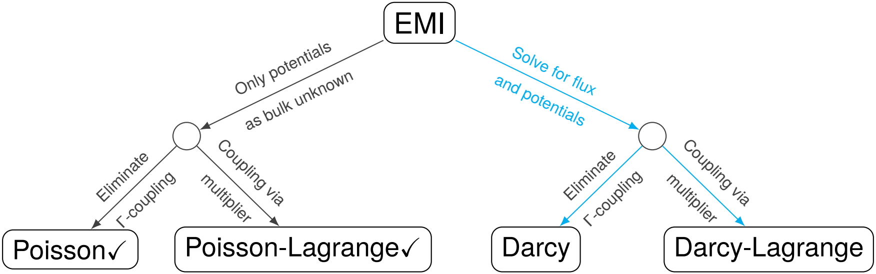
Fractional component needed for $\varepsilon \ll 1$

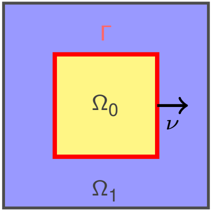




Fractional and intersection spaces yield ε-robustness

(At the price of dealing with $-\Delta_\Gamma^{-1/2}$)


EMI
Only potentials
as bulk unknown
Solve for flux
and potentials
Eliminate
Γ-coupling
Coupling via
multiplier
Eliminate
Γ-coupling
Coupling via
multiplier
Poisson✓
Poisson-Lagrange✓
Darcy
Darcy-Lagrange



$$\begin{aligned} \sigma_0 + K_0 \nabla u_0 = 0, \quad \nabla \cdot \sigma_0 = 0 \quad & \text{in } \Omega_0, \\ \sigma_1 + K_1 \nabla u_1 = 0, \quad \nabla \cdot \sigma_1 = 0 \quad & \text{in } \Omega_1, \\ \sigma_0 \cdot \nu - \sigma_1 \cdot \nu = 0 \quad & \text{on } \Gamma, \\ u_0|_\Gamma - u_1|_\Gamma - \varepsilon \sigma_0 \cdot \nu = f \quad & \text{on } \Gamma. \end{aligned}$$

$\Omega = \Omega_0 \cup \Omega_1$

- Seek flux $\sigma \in H(\text{div}, \Omega)$, $\sigma|_{\Omega_i} = \sigma_i$, $u \in L^2(\Omega)$, $u|_{\Omega_i} = u_i$
- Apply Robin in integration by parts

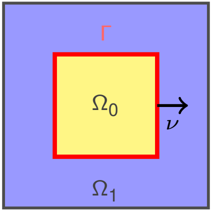
Well-posed saddle point problem for $[\sigma, u] \in H(\text{div}, \Omega) \times L^2(\Omega)$

(Mass)-conservative mixed method

$$\mathcal{A} = \begin{bmatrix} K^{-1} I + \varepsilon T_\nu' T_\nu & -\nabla \\ \nabla \cdot & \end{bmatrix}$$

$\mathcal{A} \rightarrow$ Darcy as $\varepsilon \rightarrow 0$

Könnö, J. et al (2011). Mixed FEM for problems with Robin boundary conditions. SINUM, 49(1), 285-308



$$\sigma_0 + K_0\nabla u_0 = 0,\quad \nabla\cdot\sigma_0 = 0 \qquad \text{in } \Omega_0,$$
$$\sigma_1 + K_1\nabla u_1 = 0,\quad \nabla\cdot\sigma_1 = 0 \qquad \text{in } \Omega_1,$$
$$\sigma_0\cdot\nu - \sigma_1\cdot\nu = 0 \qquad \text{on } \Gamma,$$
$$u_0|_\Gamma - u_1|_\Gamma - \varepsilon\sigma_0\cdot\nu = f \qquad \text{on } \Gamma.$$

$\Omega = \Omega_0 \cup \Omega_1$

- Seek flux $\sigma \in H(\mathrm{div},\Omega)$, $\sigma|_{\Omega_i} = \sigma_i$, $u \in L^2(\Omega)$, $u|_{\Omega_i} = u_i$
- Apply Robin in integration by parts

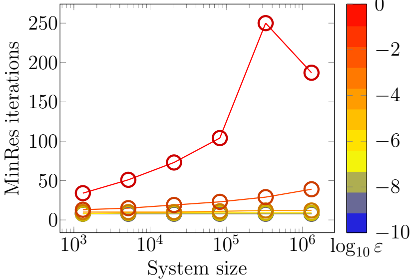
Well-posed saddle point problem for $[\sigma, u] \in H(\mathrm{div},\Omega)\times L^2(\Omega)$

(Mass)-conservative mixed method

$$\mathcal{A} = \begin{bmatrix} K^{-1}I + \varepsilon T_\nu' T_\nu & -\nabla \\ \nabla\cdot & \end{bmatrix}$$

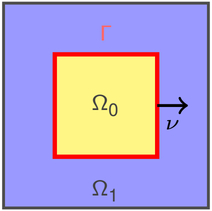
$$\mathcal{B}_0 = \begin{bmatrix} K^{-1}(I - \nabla\nabla\cdot) & \\ & KI \end{bmatrix}^{-1}$$

$\mathcal{A} \to$ Darcy as $\varepsilon \to 0$



Könnö, J. et al (2011). Mixed FEM for problems with Robin boundary conditions. SINUM, 49(1), 285-308

Darcy-like formulation - dealing with strong coupling



$$\sigma_0 + K_0\nabla u_0 = 0, \quad \nabla\cdot\sigma_0 = 0 \qquad \text{in } \Omega_0,$$
$$\sigma_1 + K_1\nabla u_1 = 0, \quad \nabla\cdot\sigma_1 = 0 \qquad \text{in } \Omega_1,$$
$$\sigma_0\cdot\nu - \sigma_1\cdot\nu = 0 \qquad \text{on } \Gamma,$$
$$u_0|_\Gamma - u_1|_\Gamma - \varepsilon\sigma_0\cdot\nu = f \qquad \text{on } \Gamma.$$

$\Omega = \Omega_0 \cup \Omega_1$

- Seek flux $\sigma \in H(\mathrm{div}, \Omega)$, $\sigma|_{\Omega_i} = \sigma_i$, $u \in L^2(\Omega)$, $u|_{\Omega_i} = u_i$
- Apply Robin in integration by parts

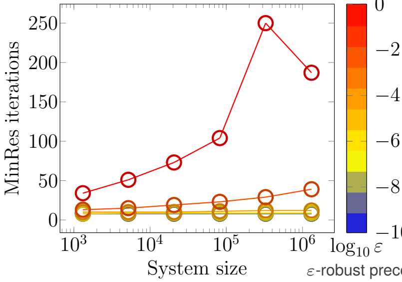
Well-posed saddle point problem for $[\sigma, u] \in H(\mathrm{div}, \Omega) \times L^2(\Omega)$

(Mass)-conservative mixed method

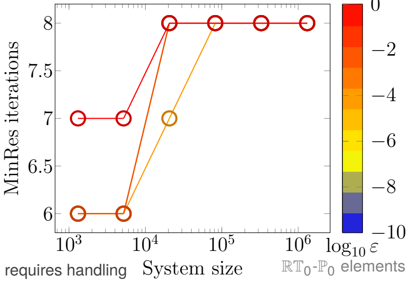
$$\mathcal{A} = \begin{bmatrix} K^{-1}I + \varepsilon T_\nu' T_\nu & -\nabla \\ \nabla\cdot & \end{bmatrix}$$

$\mathcal{A} \to$ Darcy as $\varepsilon \to 0$

$$\mathcal{B}_0 = \begin{bmatrix} K^{-1}(I - \nabla\nabla\cdot) & \\ & KI \end{bmatrix}^{-1} \qquad \mathcal{B}_1 = \begin{bmatrix} K^{-1}(I - \nabla\nabla\cdot) + \varepsilon T_\nu' T_\nu & \\ & KI \end{bmatrix}^{-1}$$



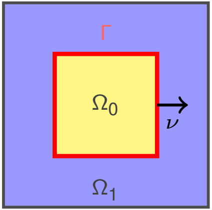
ε-robust preconditioning requires handling interface perturbation



$\mathbb{RT}_0$-$\mathbb{P}_0$ elements

Könnö, J. et al (2011). Mixed FEM for problems with Robin boundary conditions. SINUM, 49(1), 285-308

Darcy-Lagrange formulation to avoid non-standard $H(\mathrm{div})$ norm



$$\begin{aligned} \sigma_0 + K_0\nabla u_0 = 0,\quad \nabla\cdot\sigma_0 = 0 &\quad \text{in } \Omega_0,\\ \sigma_1 + K_1\nabla u_1 = 0,\quad \nabla\cdot\sigma_1 = 0 &\quad \text{in } \Omega_1,\\ \sigma_0\cdot\nu - \sigma_1\cdot\nu = 0 &\quad \text{on } \Gamma,\\ \underbrace{u_0|_\Gamma - u_1|_\Gamma}_{=\lambda} - \varepsilon\sigma_0\cdot\nu = f &\quad \text{on } \Gamma. \end{aligned}$$

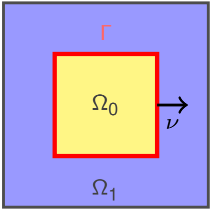
Solve explicitly for pressure jump λ, $f = f(\lambda, \cdots)$

Well-posed saddle point problem for $[\sigma, u, \lambda] \in H(\mathrm{div}, \Omega) \times L^2(\Omega) \times H^{1/2}(\Gamma)$

$$\mathcal{A} = \begin{bmatrix} K^{-1}I & -\nabla & T_\nu' \\ \nabla\cdot & & \\ T_\nu & & \varepsilon^{-1}I_\Gamma \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} K^{-1}(I - \nabla\nabla\cdot) & & \\ & KI & \\ & & \varepsilon^{-1}I_\Gamma - \overline{K}\Delta_\Gamma^{1/2} \end{bmatrix}^{-1}$$

- LM space $H^{1/2} \cap \varepsilon^{-1}L^2$
- unlike with Poisson, L^2 part can dominate for $\varepsilon \ll 1$

Darcy-Lagrange formulation to avoid non-standard $H(\mathrm{div})$ norm

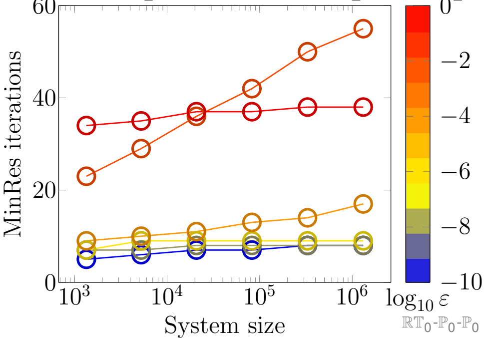


$$
\begin{aligned}
\sigma_0 + K_0 \nabla u_0 = 0, \quad \nabla\cdot\sigma_0 = 0 \quad & \text{in } \Omega_0,\\
\sigma_1 + K_1 \nabla u_1 = 0, \quad \nabla\cdot\sigma_1 = 0 \quad & \text{in } \Omega_1,\\
\sigma_0\cdot\nu - \sigma_1\cdot\nu = 0 \quad & \text{on } \Gamma,\\
\underbrace{u_0|_\Gamma - u_1|_\Gamma}_{=\lambda} - \varepsilon\sigma_0\cdot\nu = f \quad & \text{on } \Gamma.
\end{aligned}
$$

Solve explicitely for pressure jump λ, $f = f(\lambda, \cdots)$

Well-posed saddle point problem for $[\sigma, u, \lambda] \in H(\mathrm{div}, \Omega) \times L^2(\Omega) \times H^{1/2}(\Gamma)$

$$
\mathcal{A} = \begin{bmatrix} K^{-1}I & -\nabla & T'_\nu \\ \nabla\cdot & & \\ T_\nu & & \varepsilon^{-1}I_\Gamma \end{bmatrix} \qquad
\mathcal{B} = \begin{bmatrix} K^{-1}(I - \nabla\nabla\cdot) & & \\ & KI & \\ & & \varepsilon^{-1}I_\Gamma - \overline{K}\Delta_\Gamma^{1/2} \end{bmatrix}^{-1}
$$

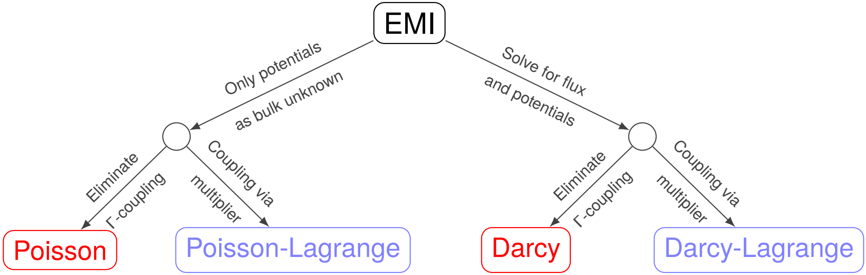


$\mathbb{RT}_0$-$\mathbb{P}_0$-$\mathbb{P}_0$ elements

- LM space $H^{1/2} \cap \varepsilon^{-1}L^2$
- unlike with Poisson, L^2 part can dominate for $\varepsilon \ll 1$

Full robustness with fractional and intersection spaces

Multiplier formulations and scaling block-diagonal preconditioners



$$\begin{bmatrix} -K_0\Delta_0 + \varepsilon^{-1}T_0'T_0 & -\varepsilon^{-1}T_0'T_1 \\ -\varepsilon^{-1}T_1'T_0 & -K_1\Delta_1 + \varepsilon^{-1}T_1'T_1 \end{bmatrix}$$

$$\begin{bmatrix} K^{-1}(I - \nabla\nabla\cdot) + \varepsilon T_\nu' T_\nu & \\ & KI \end{bmatrix}^{-1}$$

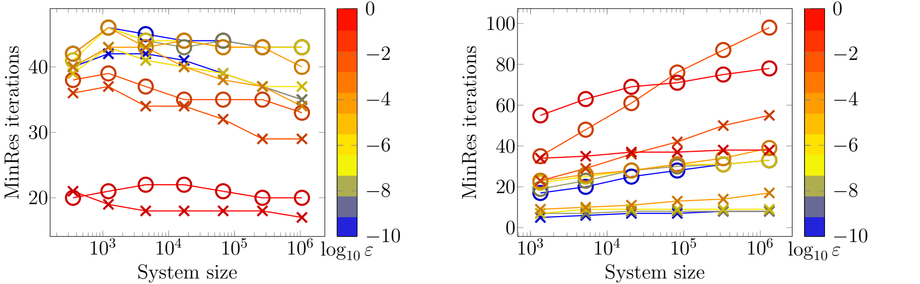
In the bulk *multiplier-free* formulations require *no-black-box solvers* for low-rank perturbations

Solvers for bulk unknowns in Lagrange formulations

Standard operators in bulk blocks in Poisson-Lagrange and Darcy-Lagrange

$$\mathcal{B}_P = \begin{bmatrix} -K_0\tilde{\Delta}_0 & & \\ & -K_1\Delta_1 & \\ & & -\overline{K}^{-1}\Delta_\Gamma^{-1/2} + \varepsilon I_\Gamma \end{bmatrix}^{-1} \quad \mathcal{B}_D = \begin{bmatrix} K^{-1}(I - \nabla\nabla\cdot) & & \\ & KI & \\ & & -\overline{K}\Delta_\Gamma^{1/2} + \varepsilon^{-1} I_\Gamma \end{bmatrix}^{-1}$$

Black-box algorithms applicable for their approximations



u-blocks by BoomerAMG

× preconditioner $\mathcal{B}_i$ realized by LU
○ bulk block in $\mathcal{B}_i$ by AMG, Γ-block by LU

σ-block by HypreAMS

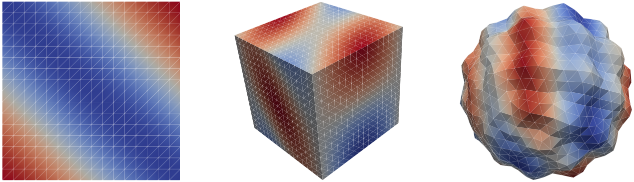
Scalabity hinges on performant solvers for (small) LM block

Kolev, T. V., Vassilevski, P. S. (2012). Parallel auxiliary space AMG solver for H(div) problems. SISC

Babuška problem $-\Delta u + u = f$ in Ω with $u = g$ in $\lambda = \partial\Omega$

$$\text{Precondition } \mathcal{A} = \begin{bmatrix} -\Delta + I & T' \\ T & 0 \end{bmatrix} \quad \text{by} \quad \mathcal{B} = \begin{bmatrix} -\Delta + I & 0 \\ 0 & (-\Delta_\Gamma)^{-1/2} \end{bmatrix}^{-1}$$

Preconditioner requires solving $\langle \underbrace{(-\Delta_\Gamma)}_{\mathsf{A}}{}^{-1/2} p, q \rangle = \langle b, q \rangle$ in Q_h



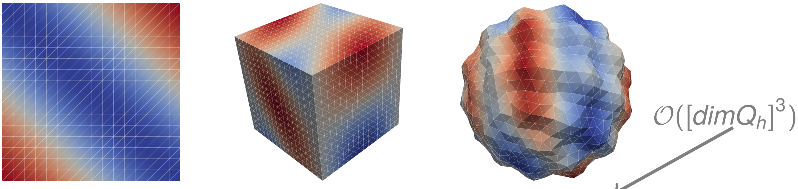
- Spectral realization $\mathsf{A}^s = (\mathsf{M}\mathsf{U})\Lambda^s(\mathsf{M}\mathsf{U})^T$, s.t. $\mathsf{A}\mathsf{U} = \mathsf{M}\mathsf{U}\Lambda$ and $\mathsf{U}^T\mathsf{M}\mathsf{U} = I$

Rational approximation for fractional order Riesz maps

Babuška problem $-\Delta u + u = f$ in Ω with $u = g$ in $\lambda = \partial\Omega$

$$\text{Precondition } \mathcal{A} = \begin{bmatrix} -\Delta + I & T' \\ T & 0 \end{bmatrix} \quad \text{by} \quad \mathcal{B} = \begin{bmatrix} -\Delta + I & 0 \\ 0 & (-\Delta_\Gamma)^{-1/2} \end{bmatrix}^{-1}$$

Preconditioner requires solving $\langle \underbrace{(-\Delta_\Gamma)}_{\mathsf{A}}{}^{-1/2} p, q \rangle = \langle b, q \rangle$ in Q_h



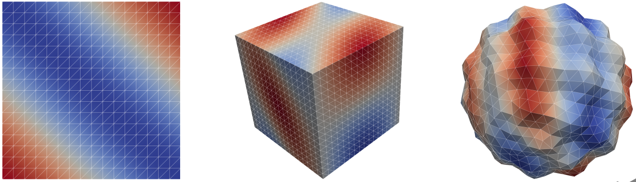
- Spectral realization $\mathsf{A}^s = (\mathsf{MU})\Lambda^s(\mathsf{MU})^T$, s.t. $\mathsf{AU} = \mathsf{MU}\Lambda$ and $\mathsf{U}^T\mathsf{MU} = I$

Rational approximation for fractional order Riesz maps

Babuška problem $-\Delta u + u = f$ in Ω with $u = g$ in $\lambda = \partial\Omega$

$$\text{Precondition } \mathcal{A} = \begin{bmatrix} -\Delta + I & T' \\ T & 0 \end{bmatrix} \quad \text{by} \quad \mathcal{B} = \begin{bmatrix} -\Delta + I & 0 \\ 0 & (-\Delta_\Gamma)^{-1/2} \end{bmatrix}^{-1}$$

Preconditioner requires solving $\langle \underbrace{(-\Delta_\Gamma)}_{\mathrm{A}}{}^{-1/2} p, q \rangle = \langle b, q \rangle$ in Q_h



$\mathcal{O}([dim Q_h]^3)$

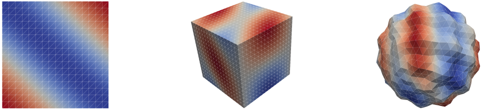
- Spectral realization $\mathrm{A}^s = (\mathrm{MU})\Lambda^s(\mathrm{MU})^T$, s.t. $\mathrm{AU} = \mathrm{MU}\Lambda$ and $\mathrm{U}^T\mathrm{MU} = I$
- Rational approximation for f, $f(x) = (x^s)^{-1}$

$$f(\mathrm{A}) = c_0 \mathrm{M}^{-1} + \sum_{i=1}^{N} (c_i \mathrm{A} + p_i \mathrm{M})^{-1}, \quad \mathbb{R} \ni p_i > 0$$

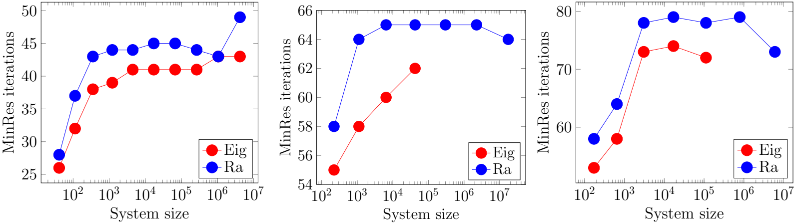
N controlled by accuracy ε_{RA} not $dim Q_h$

Budisa, A., et al. (2022). Rational approximation preconditioners for multiphysics problems. arXiv:2209.11659. Budisa, A., et al. (2022). HAZniCS–Software Components for Multiphysics Problems. arXiv:2210.13274.

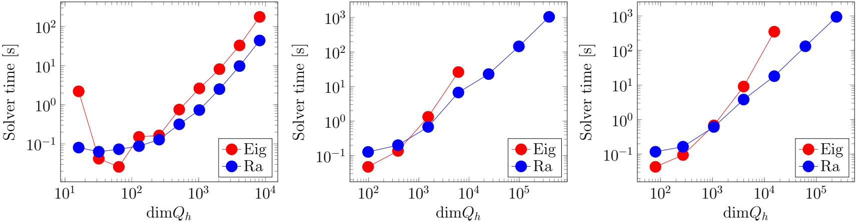
Peformance of rational approximation

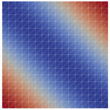


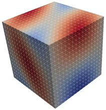
Rational approximation $\varepsilon_{\mathsf{RA}} = 10^{-12}$ leads to bounded* iterations

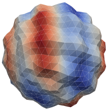


AMG for shifted Laplace problems leads to optimal complexity

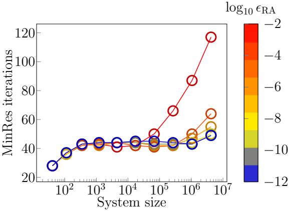


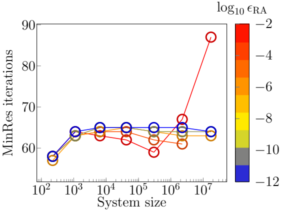


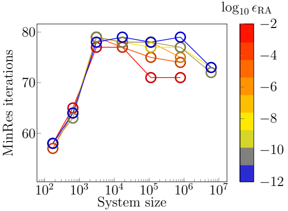




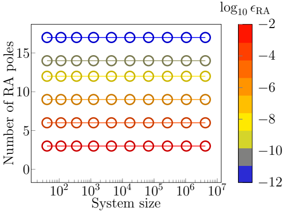
Sufficient accuracy of RA is required for stable iterations, $\varepsilon_{\mathrm{RA}} \leq 10^{-4}$

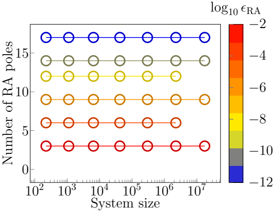


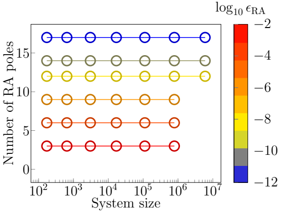




Number of RA poles grows slowly with $\varepsilon_{\mathrm{RA}}$



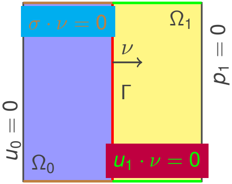




Rational approximation for sum of fractional operators

$$\begin{aligned}
-\nabla \cdot (\sigma(u_0, p_0)) &= f_0 && \text{in } \Omega_0, \\
\nabla \cdot u_0 &= 0 && \text{in } \Omega_0, \\
\sigma(u_0, p_0) &= -p_0 I + 2\mu\varepsilon(u_0) && \text{in } \Omega_0, \\
K^{-1}u_1 + \nabla p_1 &= 0 && \text{in } \Omega_1, \\
\nabla \cdot u_1 &= f_1 && \text{in } \Omega_1 \\
u_0 \cdot \nu - u_1 \cdot \nu &= 0 && \text{on } \Gamma,
\end{aligned}$$

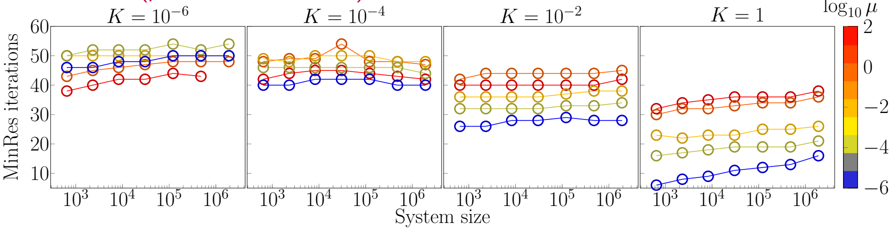
+ other interface conditions



Multiplier space $\mu^{-1/2}H^{-1/2} \cap K^{1/2}H^{1/2}$

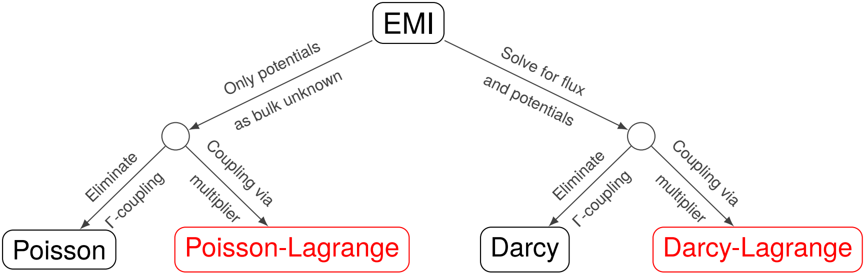
$$\mathcal{B} = \begin{bmatrix} -2\mu\nabla \cdot \varepsilon + \alpha_{\text{BJS}}\sqrt{\frac{\mu}{K}}T_0'T_0 & & & & \\ & \mu^{-1}I & & & \\ & & K^{-1}(I - \nabla(\nabla\cdot)) & & \\ & & & KI & \\ & & & & \mu^{-1}(-\Delta_\Gamma + I)^{-1/2} + K(-\Delta_\Gamma + I)^{1/2} \end{bmatrix}^{-1}$$

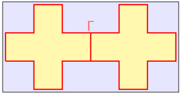
RA of $x \to (\mu^{-1}x^{-1/2} + Kx^{1/2})^{-1}$ is needed



Holter K.E., et al. (2020). Robust preconditioning of monolithically coupled multiphysics problems. arXiv:2001.05527.

Towards HPC

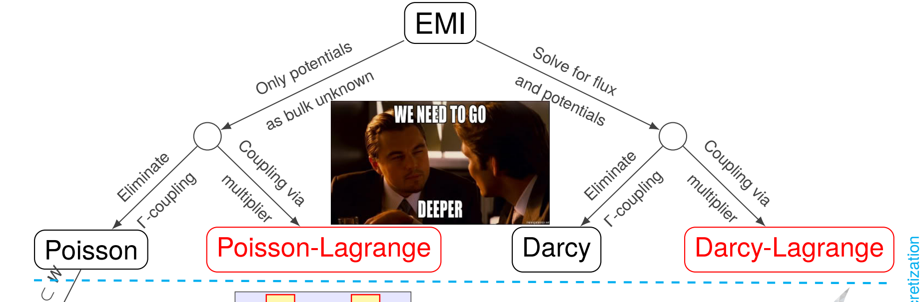




- Practical EMI applications require $-\Delta_\Gamma^{\pm 1/2}$ on non-smooth manifolds
- Lacking support in FeniCS/X (master)

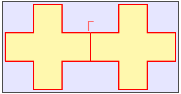
github.com/MiroK/fenics_ii

Towards HPC



$W_h \subset W$

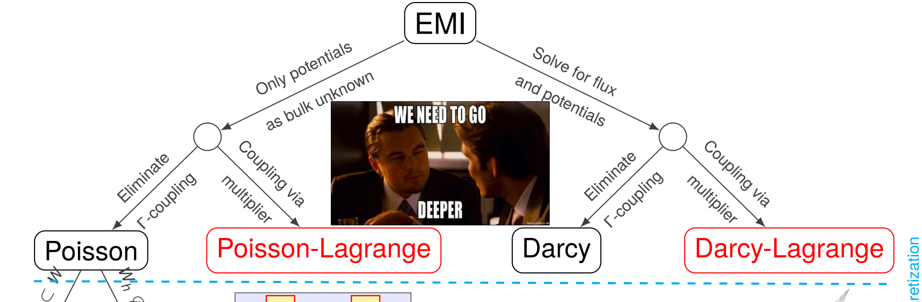
CG

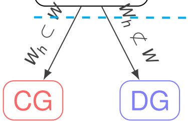


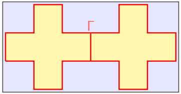
- Practical EMI applications require $-\Delta_\Gamma^{\pm 1/2}$ on non-smooth manifolds
- Lacking support in FeniCS/X (master)

github.com/MiroK/fenics_ii

Towards HPC

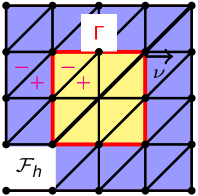






- Practical EMI applications require $-\Delta_{\Gamma}^{\pm 1/2}$ on non-smooth manifolds
- Lacking support in FeniCS/X (master)

github.com/MiroK/fenics_ii



$$
\begin{aligned}
-\nabla\cdot(K_0\nabla u_0) &= 0 \quad \text{in } \Omega_0,\\
-\nabla\cdot(K_1\nabla u_1) &= 0 \quad \text{in } \Omega_1,\\
-K_0\nabla u_0\cdot\nu + K_1\nabla u_1\cdot\nu &= 0 \quad \text{on } \Gamma,\\
\underbrace{u_0|_\Gamma - u_1|_\Gamma}_{u^+ - u^- = [\![u]\!]} + \varepsilon K_0\nabla u_0\cdot\nu &= f \quad \text{on } \Gamma.
\end{aligned}
$$

Neumann boundary $\partial\Omega_1\setminus\Gamma$

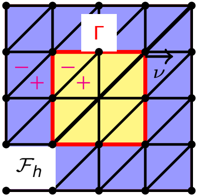
Discontinuity on *every* facet $F\in\mathcal{F}_h$

Seek $u\in W_h\not\subset W$, $u|_{\Omega_i}=u_i$ such that

$$
\sum_{T\in\Omega_h\cap\Omega_0}\int_T K_0\nabla u_0\cdot\nabla v_0 + \sum_{T\in\Omega_h\cap\Omega_1}\int_T K_1\nabla u_1\cdot\nabla v_1 + \sum_{F\in\mathcal{F}_h\cap\Gamma}\int_F \frac{1}{\varepsilon}([\![u]\!]-f)[\![v]\!]
$$

$$
-\sum_{F\in\mathcal{F}_h\setminus\Gamma}\int_F \{\!\{K\nabla v\cdot\nu\}\!\}[\![u]\!] - \sum_{F\in\mathcal{F}_h\setminus\Gamma}\int_F \{\!\{K\nabla u\cdot\nu\}\!\}[\![v]\!] + \sum_{F\in\mathcal{F}_h\setminus\Gamma}\int_F \frac{\overline{K}\gamma}{h}[\![v]\!][\![u]\!] \quad \forall v\in W_h
$$

Stabilization $\gamma>0$ set for coercivity



$$
\begin{aligned}
-\nabla \cdot (K_0 \nabla u_0) &= 0 && \text{in } \Omega_0,\\
-\nabla \cdot (K_1 \nabla u_1) &= 0 && \text{in } \Omega_1,\\
-K_0 \nabla u_0 \cdot \nu + K_1 \nabla u_1 \cdot \nu &= 0 && \text{on } \Gamma,\\
\underbrace{u_0|_\Gamma - u_1|_\Gamma}_{u^+ - u^- = [\![u]\!]} + \varepsilon K_0 \nabla u_0 \cdot \nu &= f && \text{on } \Gamma.
\end{aligned}
$$

Neumann boundary $\partial\Omega_1 \setminus \Gamma$

Discontinuity on *every* facet $F \in \mathcal{F}_h$

Seek $u \in W_h \not\subset W$, $u|_{\Omega_i} = u_i$ such that

$$
\sum_{T \in \Omega_h \cap \Omega_0} \int_T K_0 \nabla u_0 \cdot \nabla v_0 + \sum_{T \in \Omega_h \cap \Omega_1} \int_T K_1 \nabla u_1 \cdot \nabla v_1 + \sum_{F \in \mathcal{F}_h \cap \Gamma} \int_F \frac{1}{\varepsilon}([\![u]\!] - f)[\![v]\!]
$$

$$
- \sum_{F \in \mathcal{F}_h \setminus \Gamma} \int_F \{\!\{K \nabla v \cdot \nu\}\!\} [\![u]\!] - \sum_{F \in \mathcal{F}_h \setminus \Gamma} \int_F \{\!\{K \nabla u \cdot \nu\}\!\} [\![v]\!] + \sum_{F \in \mathcal{F}_h \setminus \Gamma} \int_F \frac{\overline{K}\gamma}{h} [\![v]\!][\![u]\!] \quad \forall v \in W_h
$$

Stabilization $\gamma > 0$ set for coercivity

With Γ-only-discontinous test/trial functions: *Find* $u = [u_0, u_1] \in W = H^1(\Omega_0) \times H^1(\Omega_1)$ *such that*

$$
\int_{\Omega_0} K_0 \nabla u_0 \cdot \nabla v_0 + \int_{\Omega_1} K_1 \nabla u_1 \cdot \nabla v_1 + \int_\Gamma \frac{1}{\varepsilon}([\![u]\!] - f)[\![v]\!] = 0 \quad \forall v \in W
$$

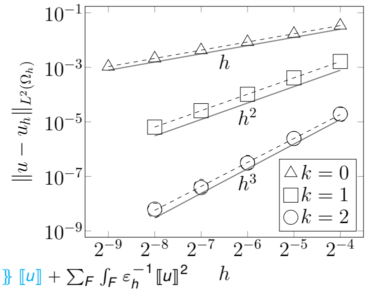
Solution operator $\mathcal{A}_h$ with familiar structure, large "interface" - $T_{i,h}$ couples on every facet

$\varepsilon_h = \varepsilon_h(\varepsilon, K, h, F)$

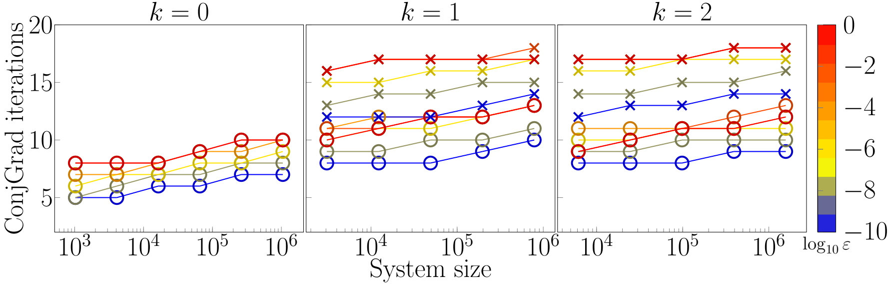
$$
\begin{bmatrix}
-K_{0,h}\Delta_{0,h} + \varepsilon_h^{-1} T'_{0,h} T_{0,h} & -\varepsilon_h^{-1} T'_{0,h} T_{1,h}\\
-\varepsilon_h^{-1} T'_{1,h} T_{0,h} & -K_{1,h}\Delta_{1,h} + \varepsilon_h^{-1} T'_{1,h} T_{1,h}
\end{bmatrix}
\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = b
$$

Discontinuous Galerkin method for EMI Poisson-formulation

- EOC for $W_h = \{v \in L^2(\Omega_h), v|_T \in \mathbb{P}^k(T)\}$
- System size $\propto \Omega_h$-element count
- conservative method (electric charge)
- runs in parallel in any FEniCS
- standard solvers seem to work



Riesz maps wrt
$\circ$ $|u|^2_{W_h} = \sum_K \int_K ||\nabla u||^2 - \sum_F \int_F \{\!\{K\nabla u \cdot \nu\}\!\} [\![u]\!] + \sum_F \int_F \varepsilon_h^{-1} [\![u]\!]^2$
$\times$ $|u|^2_{W_h} = \sum_K \int_K ||\nabla u||^2 + \sum_F \int_F \varepsilon_h^{-1} [\![u]\!]^2$

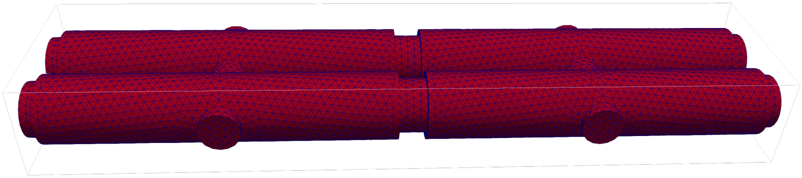


$\mathcal{B}_i$ realized by Hypre BoomerAMG V-cycle with (block) Schwarz smoother

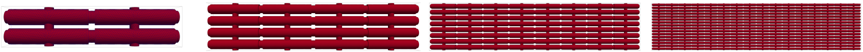
Masri, R., et al. (in prep 2023). Discontinuous Galerkin method for EMI equations

Performance of DG EMI-Poisson solver

Parametrized cylindrical "myocyte" cell defined in Gmsh



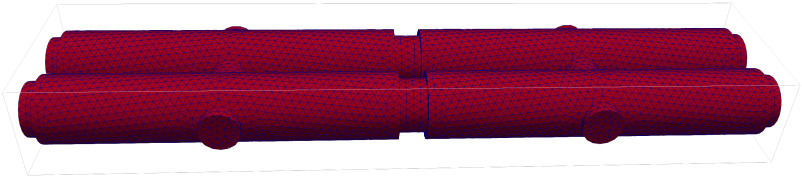
Periodicity of "tile" mesh enables building (layers of) sheets (no meshing)



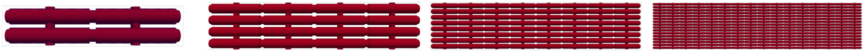
n_{EMI}	2^2	4^2	8^2	16^2	32^2	64^2	128^2	256^2
$\lvert\Omega_h\rvert/10^3$	7	26	106	423	690	6 793	27 050	108 200
T_{gen} [s]	0.004	0.006	0.01	0.04	0.16	0.65	2.7	10.5
T_{save} [s]	0.004	0.02	0.06	0.23	1.01	4.04	16.02	63.1
HDF5 [MB]	1.1	4.3	17	68	271	1100	4300	17000

$h \approx 4\mu m$

Parametrized cylindrical “myocyte” cell defined in Gmsh



Periodicity of “tile” mesh enables building (layers of) sheets (no meshing)



n_{EMI}	2^2	4^2	8^2	16^2	32^2	64^2	128^2	256^2
$\lvert\Omega_h\rvert/10^3$	7	26	106	423	690	6 793	27 050	108 200
T_{gen} [s]	0.004	0.006	0.01	0.04	0.16	0.65	2.7	10.5
T_{save} [s]	0.004	0.02	0.06	0.23	1.01	4.04	16.02	63.1
HDF5 [MB]	1.1	4.3	17	68	271	1100	4300	17000

$h \approx 4\mu m$

Preliminary scaling: 32procs, 128GB RAM $\mathbb{P}^{\text{disc}}{}_1$ elements

n_{EMI}	2^2	4^2	8^2	16^2	32^2	64^2
$\dim W_h/10^3$	26	106	423	1 690	6 762	27 049
n_{iters}	47	48	48	48	47	47
T_{KSP} [s]	0.12	0.35	145	155	185	323

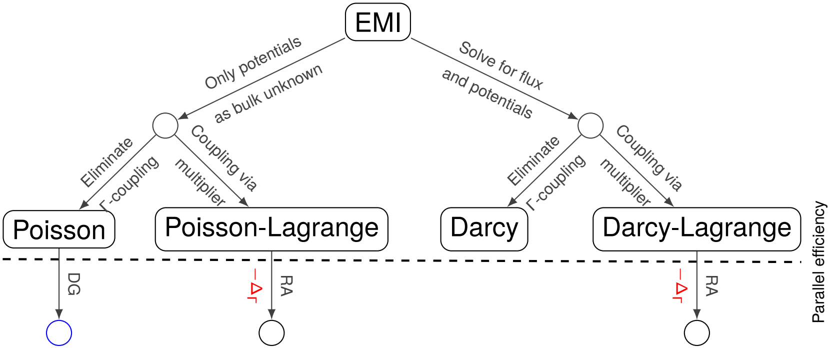
KSP=$\mathcal{B}$-setup+ConjGrad

$\dim W_h/10^3$	n_{iters}	T_{KSP} [s]
3 588	58	194
6 762	47	185
27 576	34	298

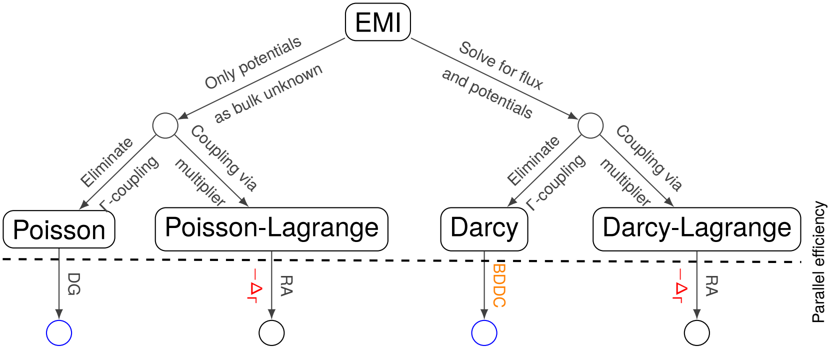
BoomerAMG with point smoother
relative tolerance 10^{-8}

Conclusions and future work

- “Balance” the EMI solver tree



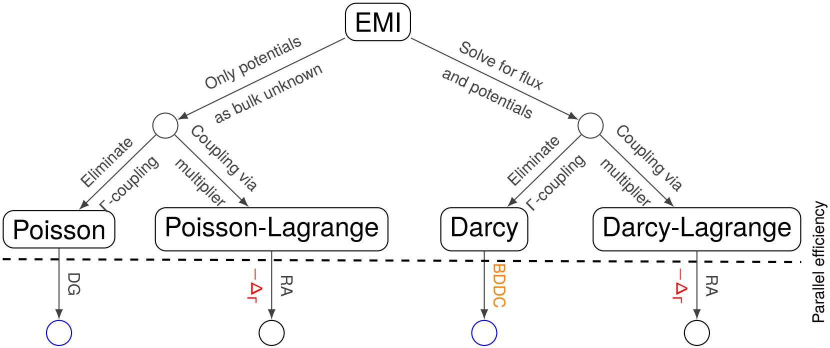
- “Balance” the EMI solver tree



Zampini, S., Tu, X. (2017). Multilevel BDDC · for flow in porous media. SISC

Conclusions and future work

- “Balance” the EMI solver tree

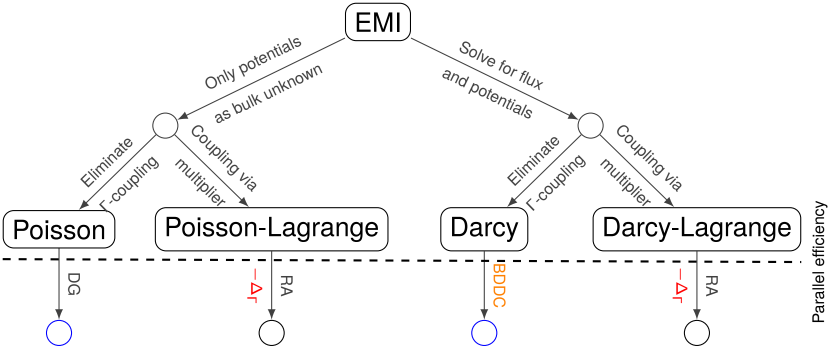


Zampini, S., Tu, X. (2017). Multilevel BDDC · for flow in porous media. SISC

- How critical is parameter robustness in practice?
- (DG) solvers in real applications
- Coupling EMI and mechanics

Conclusions and future work

- “Balance” the EMI solver tree



Zampini, S., Tu, X. (2017). Multilevel BDDC · for flow in porous media. SISC

- How critical is parameter robustness in practice?
- (DG) solvers in real applications
- Coupling EMI and mechanics

Thank you for your attention!

Ana Budiša (Simula)
Ada Johanne Ellingsrud (Simula)
Kent-Andre Mardal (UiO)
Rami Masri (Simula)
Stefano Zampini (Kaust)
Ludmil T. Zikatanov (Penn State)